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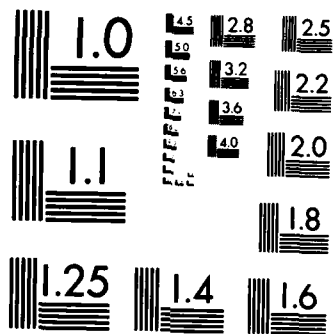
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MRC Technical Summary Report #2943

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

The derivation of the stress-integral form of the Higashitani-Pritchard-Baird-Lodge relation cannot be applied to plane Poiseuille flow. It ignores the contribution of the streamwise pressure gradient, and doing so can lead to contradictions. Furthermore, the variable change which leads to the stress-integral form of the HPBL relation is not valid in a second-order fluid approximation for slow flows. Correcting these deficiencies leads to a modified relation which appears intractable. Nevertheless, the Tanner-Pipkin result that $P_e = N_1/4$ in a second order fluid is valid when properly interpreted.

AMS (MOS) Subject Classification: 76A10

Key Words: Hole-pressure, Poiseuille flow, Creeping flow, Second-order fluid, Streamline co-ordinates, Maxwell fluid, Hydrostatic pressure, Thrust, First normal-stress difference, Non-Newtonian, Viscometric flow, Johnson-Segalman fluid

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ON HOLE-PRESSURES IN PLANE POISEUILLE FLOW OVER TRANSVERSE SLOTS

D. S. Malkus and Minwu Yao

1. Introduction

The symbols used in this paper are defined in Appendix I at the end of main text. The problem in question is plane, creeping flow over a transverse slot, which is illustrated in Figure 1 and described in refs. 1-4. Ref. 1 describes a proposed relation between pressure differences, P_e , and the first normal-stress difference, N_1 . It is clear that ref. 1 is primarily concerned with Couette flow (plate-driven) and not Poiseuille flow (pressure-gradient driven). In refs. 2-8 the possibility of applying the relation to Poiseuille flow has been explored experimentally and numerically. The final relation is attributable to Higashitani, Pritchard, Baird, and Lodge [2,3], and has two forms for creeping flow:

$$P_e = \int_0^{\sigma_w} \frac{N_1}{2\tau} d\tau \quad (1a)$$

$$N_{1w} = 2 \frac{d \ln P_e}{d \ln \sigma_w} P_e \quad (1b)$$

where σ_w and N_{1w} are wall values of the primary viscometric functions, and P_e is the pressure difference from top to bottom of the slot. Eq. (1b) is easily derived from eq. (1a), and the purpose of this paper is to show that the derivation of eq. (1a) is, at best, inapplicable to plane Poiseuille flow. It is with some irony, then, that the authors can state that numerical results seem to corroborate eqs. (1a) and (1b) to within about 10%, at worst, for a Maxwell fluid and to validate eq. (1b) to "working rheological accuracy" for some more realistic fluids. These results will be discussed thoroughly elsewhere [7,8]. It seems to be an unavoidable conclusion, however, that the numerical and experimental results owe their pleasing concordance to some other cause than the validity of the stress-integral form of the HPBL relation eq. (1a), at least as it is currently understood.

This paper does not directly concern numerical results, but it does often rely on evidence which can be observed from the relatively simple problem of plane Stokes flow over a transverse slot. Since the computing technology at the time of this writing allows many researchers in fluid mechanics to obtain and analyze solutions to Stokes flow problems with relative ease, rather than tabulating data to support our statements about Stokes flow, we invite the skeptical reader to verify them directly. We believe that most of our claims about Stokes flow are reasonable, if not obvious, and should not provoke controversy. The

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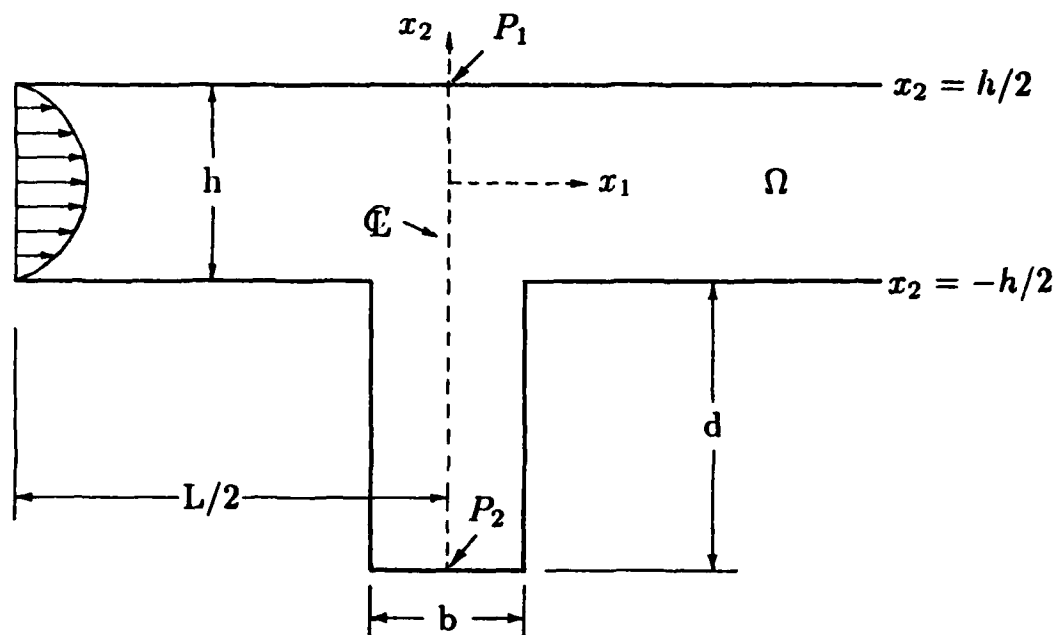


FIGURE 1

Domain and dimensions for plane flow over a transverse slot. Rheological, Cartesian co-ordinates, x_i , have origin at intersection of slot \mathbb{C} and centerline of channel or 'die.'

reason that Stokes flow is central to what we present here is that our observations are based on, but not confined to, the second-order fluid range of non-Newtonian behavior.

2. Preliminaries

It will serve the purposes of completeness and facilitate the comparison of the theoretical results obtained here with numerical results discussed elsewhere if we begin by deriving the equation for plane Poiseuille flow, taking particular care to fix the definition of the hydrostatic pressure function in the same manner as in the numerical model of [4]. This also fixes the arbitrary constant in p .

2.1 Pressure definition:

In plane flow, the pressure is defined so that

$$\sigma = \begin{bmatrix} N_1/2 & \sigma_{12} & 0 \\ \sigma_{12} & -N_1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - p\mathbf{I} \quad (2)$$

where $N_1 = \sigma_{11} - \sigma_{22}$. In a shearing flow, N_1 is the usual viscometric function. Eq. (2) implies that

$$p \equiv -(\sigma_{11} + \sigma_{22})/2 \quad (3)$$

In addition, we specify that

$$\int_{\Omega} p dV = 0 \quad (4)$$

The equations of motion for undisturbed flow ($d = 0$),

$$\begin{aligned} \sigma_{11,1} + \sigma_{12,2} &= 0 \\ \sigma_{21,1} + \sigma_{22,2} &= 0 \end{aligned} \quad (5)$$

imply that for plane Poiseuille flow

$$\begin{aligned} \sigma_{12,2} &= p_{,1} - \dot{\gamma}^0 (N_1/2)_{,1} = k \\ \sigma_{22,1} - (N_1/2)_{,2} &= p_{,2} \end{aligned} \quad (6)$$

or, imposing a boundary condition at the wall and symmetry condition at mid-channel

$$\begin{cases} \sigma_{12} = kx_2 \\ u_1(\cdot, h/2) = 0 \\ u_{1,2}(\cdot, 0) = 0 \end{cases}$$

where $\sigma_{12} = \eta \dot{\gamma}$ and $\dot{\gamma}(0) = 0$. Integrating eqs. (6) implies

$$p = kx_1 + f(x_2)$$

$$p = -N_1/2 + g(x_1)$$

thus

$$kx_1 + N_1(x_2)/2 + f(x_2) - g(x_1) = 0$$

and

$$p = kx_1 - N_1/2 + C \quad (7)$$

From eq. (4)

$$C = \frac{1}{2h} \int_{-h/2}^{h/2} N_1(x_2) dx_2$$

and for a Maxwell fluid, an Oldroyd B fluid, any fluid for which $N_1 = \nu_1 \dot{\gamma}^2$, $\nu_1 = \text{constant}$, and $\eta = \text{constant}$,

$$C = N_{1w}/6 \quad (8)$$

Eq. (8) holds approximately for any fluid which has a second-order fluid range, when the flow is sufficiently slow. For such fluids and/or flows and the above, we may deduce the form of the stress tensor in undisturbed plane Poiseuille flow:

$$\sigma = \begin{bmatrix} -kx_1 + N_1 - N_{1w}/6 & \sigma_w & 0 \\ \sigma_w & -kx_1 - N_{1w}/6 & 0 \\ 0 & 0 & -kx_1 + N_1/2 - N_{1w}/6 \end{bmatrix} \quad (9)$$

It should be kept in mind that N_{1w} is a constant. Observe that $\sigma_{12} = \sigma_w$ at the top wall, as illustrated in Figure 2. For the pictured flow direction $\sigma_w < 0$. A more crucial observation is that σ_{22} is a linear function of x_1 alone and has slope $-k$; this will soon become important to our arguments. σ_{11} varies across channel as a function of N_1 .

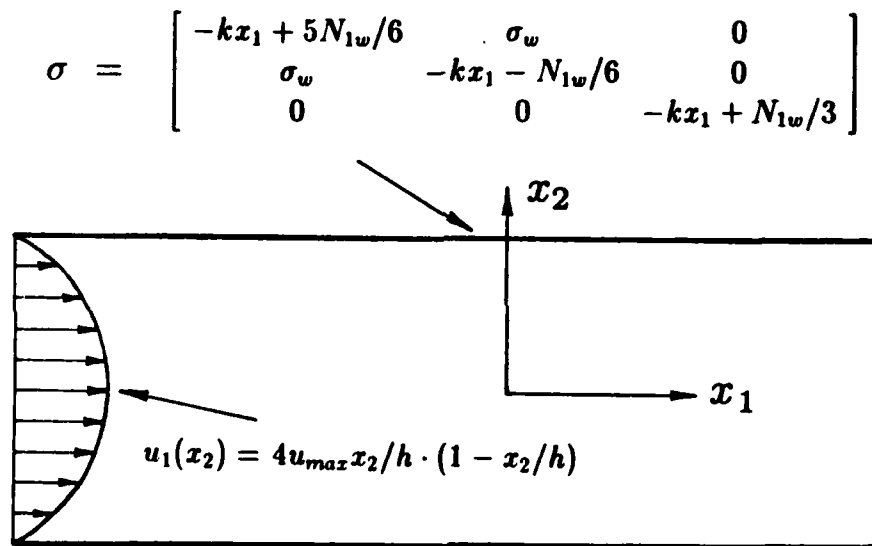


FIGURE 2

Undisturbed plane Poiseuille flow, showing stress tensor at the wall where $\sigma_{12} = +\sigma_w$ (a negative value for the pictured flow). $\sigma_{12} = -\sigma_w$ at the opposite wall, and the stress tensor is otherwise the same. Can be thought of as slot-flow with a $d = 0$ slot.

3. Tanner-Pipkin Results

We shall primarily be concerned with slow flows, in which for many constitutive equations, the second-order expansion is valid [9,10]. In such cases we presume that by solving the steady plane flow problem described here with the truncated "second-order fluid" constitutive equation [9,10] we produce a solution very close to that obtained when the full constitutive equation is used (when the flow is sufficiently slow). By the Tanner-Giesekus theorem, the Stokes solution for the velocities, u , with a modified pressure, p , satisfies the second-order fluid equations [9,10]. The resulting stress field is

$$\sigma = -[\bar{p} - T(u_1 \bar{p}_{,1} + u_2 \bar{p}_{,2})] \mathbf{I} + \eta \mathbf{A} - \eta T(\mathbf{B} - \mathbf{A}^2 - \frac{1}{2} \dot{\gamma}^2 \mathbf{I}) + \eta T^*(\mathbf{A}^2 - \dot{\gamma}^2 \mathbf{I}) \quad (10)$$

where \mathbf{A} and \mathbf{B} are the first two Rivlin-Ericksen tensors; η , T and T^* are the constants of the second-order fluid expansion, \bar{p} is the Stokes-flow pressure field, and $\dot{\gamma}^2 = \text{tr} \mathbf{A}^2/2$.

$$\mathbf{A} = \begin{bmatrix} 2u_{1,1} & u_{1,2} + u_{2,1} \\ u_{1,2} + u_{2,1} & -2u_{1,1} \end{bmatrix}$$

The symmetries of the Stokes solution imply that u_1 is even and u_2 is odd about \mathcal{C} as functions of x_1 . Thus $u_{1,1} = u_{2,2} = 0$ on \mathcal{C} ($x_1 = 0$) and

$$\mathbf{A} = \begin{bmatrix} 0 & \dot{\gamma}(x_2) \\ \dot{\gamma}(x_2) & 0 \end{bmatrix} \quad (11)$$

Therefore

$$\mathbf{A}^2 = \dot{\gamma}^2 \mathbf{I}$$

and the T^* -term in eq. (10) vanishes on the slot centerline for a second-order fluid. The ηT -term is

$$-\eta T(\mathbf{B} - \frac{3}{2} \dot{\gamma}^2 \mathbf{I})$$

and

$$\mathbf{B} = \frac{D}{Dt} \mathbf{A} + \mathbf{A} \nabla \mathbf{u} + (\nabla \mathbf{u})^T \mathbf{A}$$

Now

$$\begin{aligned} \frac{D\mathbf{A}}{Dt} &= [u_1 A_{ij,1} + u_2 A_{ij,2}] \\ &= u_1 \begin{bmatrix} 2u_{1,11} & (u_{1,2} + u_{2,1})_{,1} \\ (u_{1,2} + u_{2,1})_{,1} & -2u_{1,11} \end{bmatrix} \end{aligned}$$

We observe first that $u_{1,21} = u_{1,12} = 0$, and since u_2 is an odd function about \mathcal{C} , $u_{2,11} = 0$, and

$$\frac{D\mathbf{A}}{Dt} = u_1 \begin{bmatrix} 2u_{1,11} & 0 \\ 0 & -2u_{1,11} \end{bmatrix} \quad (12)$$

We do not believe that it can be argued that $u_{1,11}$ is zero on \mathcal{C} . It may be that $u_{1,11}$ is "small" (numerical evidence suggests this), but we do not believe it can be entirely ignored; we know that $u_{1,1} = u_{1,111} = 0$, and thus

$$u_1(\delta x_1, x_2) \cong u_1(0, x_2) - \frac{1}{2}(\delta x_1)^2 u_{1,11} - \frac{1}{24}(\delta x_1)^4 u_{1,1111}$$

Were $u_{1,11}$ zero or negligibly small, this would imply a flatness to the streamlines near \mathcal{C} which does not seem to correspond to experiment or computation.

So we retain $u_{1,11}$ and observe that

$$\mathbf{A} \nabla \mathbf{u} = \begin{bmatrix} 0 & \dot{\gamma} \\ \dot{\gamma} & 0 \end{bmatrix} \begin{bmatrix} 0 & \dot{\gamma} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \dot{\gamma}^2 \end{bmatrix}$$

and thus on \mathcal{C}

$$\begin{aligned} \boldsymbol{\sigma} = & -(\bar{p} - T u_1 \bar{p}_{,1}) \mathbf{I} + \eta \begin{bmatrix} 0 & \dot{\gamma} \\ \dot{\gamma} & 0 \end{bmatrix} \\ & - \eta T \left(u_1 \begin{bmatrix} 2u_{1,11} & 0 \\ 0 & -2u_{1,11} \end{bmatrix} + \begin{bmatrix} -\frac{3}{2}\dot{\gamma}^2 & 0 \\ 0 & \frac{1}{2}\dot{\gamma}^2 \end{bmatrix} \right) \end{aligned} \quad (13)$$

From eq. (13), we may draw several important conclusions:

I. On \mathcal{C}

$$\begin{cases} \sigma_{12} = \eta \dot{\gamma} \\ \sigma_{11} - \sigma_{22} = 2\eta T(\dot{\gamma}^2 - 2u_1 u_{1,11}) \end{cases} \quad (14)$$

II.

$$P_e = -\bar{p} + \frac{1}{2}\eta T(\dot{\gamma}_w^0)^2 - \bar{p} = N_{1w}^0/4 \quad (15)$$

where estimate II applies when d is such that $u_1 = 0$ or $u_{1,11} = 0$ ($u_1 = 0$ at the top of the die always). This will be the case for $d = \infty$ and approximately so for $d \gg b$. Numerical results suggest that $d/b \geq 1$ is enough to make $P_e = N_{1w}^0/4$ a good approximation, and this in turn is close to $N_{1w}/4$ of undisturbed flow if $h/b \geq 2$ [4,11].

4. Consequences of Tanner-Pipkin Results

First we observe that some of the previous developments seem to correspond very closely to computed results in slow flows over transverse slots. P_1 is substantial and positive, in spite of the disturbance induced by the slot, if $h/b = 2$ and $d/b = 4$. In Table 1, computations with a Maxwell fluid at $De = 0.25$ are tabulated. We find that $P_1 \cong N_{1w}^0/6$ (see Figure 3). This is slightly lower than $N_{1w}/6$, but still significant. Evidently the addition of the slot cannot have contributed much to $\int_{\Omega} p dV$, since we have nearly the same thrust, $P \equiv -\sigma_{22}$, at the top of the die as we would have without the slot. The only

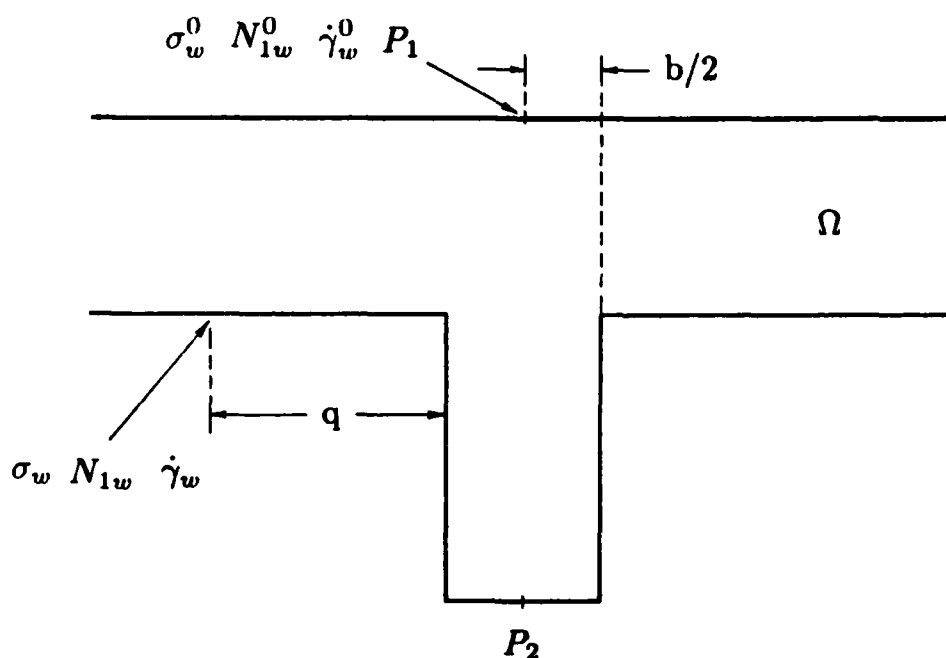


FIGURE 3
Location of undisturbed and disturbed values (indicated by superscript zero) of quantities important to the present work.

D_e	$-\sigma_w$	$-\sigma_w^0$	$N_{1w}/6$	$N_{1w}^0/6$	P_1
0.25	1062	1008	88.33	80.00	81.12

TABLE 1
Disturbed and undisturbed values of quantities in Figure 3 for a Maxwell fluid at a low D_e (probably in the second-order fluid range [7]).

effect the small disturbance at the opposite wall seems to have is to lower the shear-rate, and thus N_1 , at the opposite wall. The only effect this has on P there is to lower it to one sixth of N_{1w}^0 , as it would be in undisturbed Poiseuille flow with the slightly lower shear-rate.

Table 2 shows the same calculations with a modified Johnson-Segalman fluid [12] at $D_e = 0.24$. N_{1w} and σ_w differ so significantly from the Maxwell case, because a large retardation time, Λ , necessitates a higher shear rate to achieve a similar $D_e = (T - \Lambda)\dot{\gamma}_w$.

D_e	$-\sigma_w$	$-\sigma_w^0$	$N_{1w}/6$	$N_{1w}^0/6$	P_1
0.24	1530	1459	122.4	111.4	111.2

TABLE 2
Disturbed and undisturbed values of quantities in Figure 3 for
a modified Johnson-Segalman [12] fluid at a low D_e .

Here, too, we find that $P_1 \cong N_{1w}^0/6$.

For $h/b \geq 2$, the disturbance due to the slot has the effect of deforming the streamlines in Stokes flow downward towards the slot mouth slightly (see Figure 4 of ref. 12, for example). An effect of this is that, while $\sigma_{12}(0,0) = 0$ in undisturbed flow, $\sigma_{12}(0, x'_2) = 0$ in disturbed flow, for $x'_2 < 0$ which has small magnitude with respect to h . Next we observe that if h/b is sufficiently large, the disturbance to channel flow induced by the slot will not be of sufficient intensity to disturb $\sigma_{22}(0, x'_2)$ significantly from the value

$$\sigma_{22}(0, h/2) = -P_1 \cong -N_{1w}^0/6$$

We note that in undisturbed plane Poiseuille flow, $\sigma_{22}(x_1, x_2)$ is constant in x_2 . On the other hand, by Tanner and Pipkin's results $P_1 - P_2 = N_{1w}^0/4$, and thus $-P_2 = \sigma_{22}(0, -d - h/2) = N_{1w}^0/12$. A value near this is confirmed numerically for $0 \leq D_e \leq 0.3$. Putting this all together, we see that $\sigma_{22}(0, x_2)$ changes sign from negative to positive as we move from the wall opposing the slot towards the bottom of the slot, and it changes sign on the slot side of the channel centerline, at a point significantly closer to the slot than $(0, x'_2)$. In sum, observation of Stokes flow solutions, the implications of these observations in eq. (13), and the Tanner-Pipkin hole-pressure result lead us to deduce three important qualitative features of the second-order fluid flow (at least for sufficiently large h/b), illustrated in Figure 4:

- (1) The existence of $x'_2 < 0$ with small magnitude with respect to h .
- (2) $\sigma_{22}(0, x'_2)$ is significantly negative.
- (3) $\sigma_{22}(0, -d - h/2)$ is significantly positive.

5. HPBL in Stress Integral Form

We refer to eq. (1a) as the stress integral form of the HPBL relation (creeping flow case). From the discussion above and the derivation of ref. 1, it is clear that a more precise form of the relation should take the disturbance at the wall opposing the slot into account. This results in

$$P_e = \int_0^{\sigma_w^0} \frac{N_1}{2\tau} d\tau \quad (16)$$

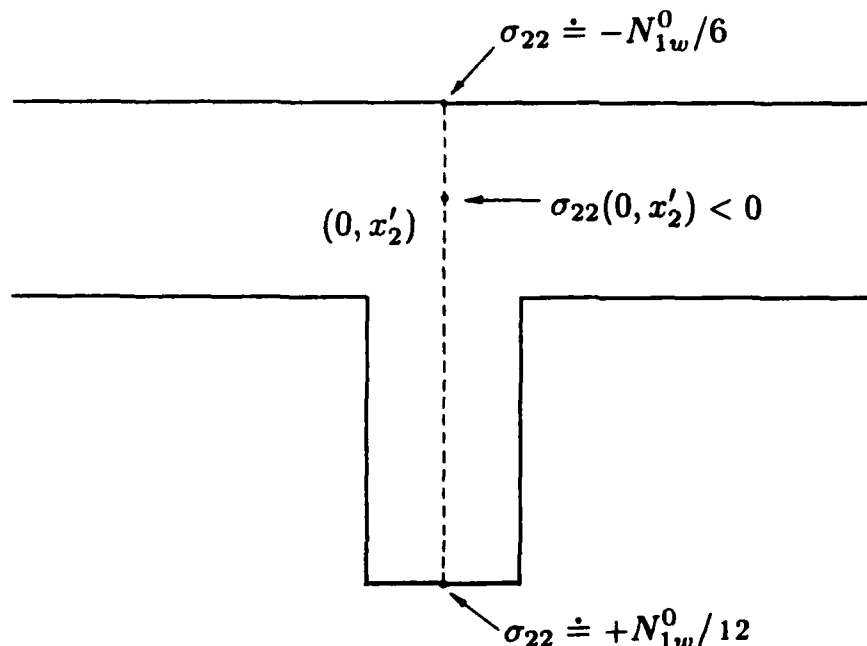


FIGURE 4

Qualitative picture of flow over a transverse slot in second-order fluid range. Important features are (a) that σ_{22} is significantly negative at mid-channel and in the upper half of the channel, but (b) σ_{22} does change sign somewhere below mid-channel and above the bottom of the slot.

We should emphasize that our experience, guided by experimental and numerical evidence [2,3,6-8], is that eq. (16) is a remarkably accurate prediction of P_e , even in a range of flow-rates far beyond the slow flows considered here. We contend, however, that there are several puzzling flaws in the derivation of eq. (16), at least when applied to Poiseuille flow.

5.1 Observation on HPBL derivation:

The basic form derived by Higashitani and Pritchard is

$$P_e = \int_{y_2(-d-h/2)}^{y_2(h/2)} \frac{N_1}{2\tau} \frac{\partial \tau}{\partial y_2} dy_2 \quad (17)$$

where " y_2 " is a streamline coordinate (taken as a function of the Cartesian coordinates in the integration limits). The assumptions under which the derivation is carried out (see

Appendix II) imply the limits may be any stream function values on \mathcal{Q} and

$$\begin{aligned}\sigma_{22}(0, x_2^b) - \sigma_{22}(0, x_2^a) &= -\frac{1}{2} \int_{y_2^a}^{y_2^b} \frac{N_1}{\tau} \frac{\partial \tau}{\partial y_2} dy_2 \\ &= -\frac{1}{2} \int_{\sigma_{12}(0, x_2^a)}^{\sigma_{12}(0, x_2^b)} \frac{N_1}{\tau} d\tau\end{aligned}\quad (18)$$

where $x_2^a = x_2(y_2^a)$ and $x_2^b = x_2(y_2^b)$.

5.2 Three Flaws in HPBL

Flaw A: *Piecewise application is contradictory.* Eq. (18) is inconsistent with the qualitative picture of the observed features, (1)–(3), illustrated in Figure 4, which may be further abstracted from any specific numerical results, as follows

Assume:

- (i) The existence of x_2' , where $\sigma_{12}(0, x_2') = 0$.
- (ii) $\sigma_{22}(0, h/2)$ has some fixed value and $\sigma_{22}(0, h/2) \cdot \sigma_{22}(0, x_2') > 0$.
- (iii) $\sigma_{22}(0, h/2) \cdot \sigma_{22}(0, -d - h/2) < 0$.

Then rechanging the variable in eq. (16)

$$\begin{aligned}P_e &= \int_0^{\sigma_{12}^u} \frac{N_1}{2\tau} d\tau = \int_{y_2(-d-h/2)}^{y_2(h/2)} \frac{N_1}{2\tau} \frac{\partial \tau}{\partial y_2} dy_2 \\ &= \int_{y_2(x_2')}^{y_2(h/2)} \frac{N_1}{2\tau} \frac{\partial \tau}{\partial y_2} dy_2 = \sigma_{22}(0, x_2') - \sigma_{22}(0, h/2)\end{aligned}\quad (19)$$

The crucial step follows from the fact that the variable change can make no distinction between $y_2(-d - h/2)$ and $y_2(x_2')$, since at both locations $\sigma_{12} = 0$. But the qualitative picture abstracted in (i)–(iii) yields

$$|P_e| = |\sigma_{22}(0, h/2) - \sigma_{22}(0, -d - h/2)| > |\sigma_{22}(0, h/2) - \sigma_{22}(0, x_2')|.$$

The contradiction lies in the fact that the inequality is strict by (i)–(iii), yet eq. (19) argues both sides should be equal. In fact, on the basis of (1)–(3), one might expect that the inequality should be satisfied by a wide margin, since on the left, the stresses add, but on the right, two nearly equal stresses are subtracted. Numerical evidence shows that the two sides of the above inequality are not close.

Flaw B: $\sigma_{11} - \sigma_{22}$ is not viscometric on \mathcal{Q} . This seems to invalidate the change of variable required to derive eq. (16) from eq. (17) and perhaps more disturbing, implies that the integral in eq. (17) may not even exist. The change of variable requires that N_1 can be written as a unique function of $\tau = \sigma_{12}(x_2)$. This is not possible in general as eq. (14)

shows that $\sigma_{11} - \sigma_{22}$ is multiple-valued as a function of σ_{12} on \mathcal{C} . The multiple-valuedness can be seen as follows:

Observation: Consider that point $x'_2 \in (-h/2, h/2)$ where $\sigma_{12} = 0$. There $u_1(x'_2) \neq 0$, and $u_{1,11}(x'_2) \neq 0$, and thus at x'_2 ,

$$\dot{\gamma} = 0; \quad \frac{\sigma_{11} - \sigma_{22}}{2\eta T} = -2u_1 u_{1,11} \neq 0$$

On the other hand, for a deep enough hole, $\sigma_{11} - \sigma_{22} \rightarrow 0$ with $\sigma_{12} \rightarrow 0$. We thus have

$$\sigma_{11} - \sigma_{22} = \begin{cases} -4\eta T u_1 u_{1,11} \neq 0, & \text{at } \sigma_{12} = 0; \\ 0, & \text{also at } \sigma_{12} = 0. \end{cases} \quad (20)$$

Numerical results strongly support these conclusions. In fact, at $(0, x'_2)$ where $\sigma_{12} = 0$, and $\sigma_{11} - \sigma_{22} \neq 0$, there is a pole in $(\sigma_{11} - \sigma_{22})/\tau$. These poles have been observed experimentally by D. G. Baird [3]. Note that we have refrained from referring to $\sigma_{11} - \sigma_{22}$ as " N_1 ," reserving that notation for the viscometric function. What we are saying here is that, on \mathcal{C} , the stress-difference is *not* the viscometric function it is presumed to be by the HPBL derivation, because the flow is evidently not one of constant curvilinear shearing history. Thus $\sigma_{11} - \sigma_{22}$ does not take on the values it would have in a viscometric flow with the same \mathbf{A} which occurs at any given point on the \mathcal{C} . Another way of saying this is that in the \mathcal{C} flow \mathbf{A} is of viscometric form, but $\frac{D\mathbf{A}}{Dt}$ is not zero, as it would be in a viscometric flow. Evidence suggests that $u_1 u_{1,11}$ is quite small in most flows where the assumptions about \mathbf{A} on the \mathcal{C} are approximately valid. The troublesome problem is that, even if $u_1 u_{1,11}$ is very small, whenever it is not identically zero, but $\dot{\gamma} = 0$, the centerline integral of HPBL does not seem to exist.

But in Poiseuille flow there appears to be an even more serious flaw. Discussion of Flaw C is prefaced by the observation that the assumptions of HPBL should apply to undisturbed Poiseuille flow ($d = 0$). In this case, $u_1 u_{1,11} \neq 0$ is not a problem, and N_1 and σ_{12} have viscometric values, but HPBL is contradictory: by eq.(16)

$$\sigma_{22}(0, h/2) - \sigma_{22}(0, 0) = \int_0^{\sigma_{12}} \frac{N_1}{2\tau} d\tau = -\frac{N_1}{4} \quad (21)$$

but this thrust difference should clearly be zero. Technically, Higashitani and Pritchard's derivation reduces to " $0 = 0$ " for undisturbed flow and needs a more complicated argument to cover the case in which the streamlines are not curved. However, we believe that this problem is only technical, and the real problem is that HPBL assumes away the streamwise gradient, k .

Flaw C: *The derivation ignores the driving gradient.* The equation in streamline co-ordinates which is integrated from point x_2^a to x_2^b to give the thrust difference is not

$$\frac{\partial S_{12}}{\partial y_2} \rightarrow \frac{2S_{12}}{S_{11} - S_{22}} \frac{\partial S_{22}}{\partial y_2} = 0 \quad (22)$$

but

$$\frac{1}{h_1} \frac{\partial S_{11}}{\partial y_1} + \frac{1}{h_2} \frac{\partial S_{12}}{\partial y_2} + \frac{2}{h_2} \frac{S_{12}}{S_{11} - S_{22}} \frac{\partial S_{22}}{\partial y_2} = 0 \quad (23)$$

where S_{ij} are physical components of stress in streamline coordinates. This follows from the geometric assumptions of Higashitani and Pritchard, which imply h_1 and $h_2 = \text{unknown}$, $h_3 = 1$ and $\frac{\partial h_i}{\partial y_j} = 0$ unless $i = 1$ and $j = 2$; and

$$\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} = \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} = 0$$

and

$$\left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} = -\frac{h_1}{h_2^2} \frac{\partial h_1}{\partial y_2},$$

$$\left\{ \begin{matrix} 1 \\ 21 \end{matrix} \right\} = \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} = \frac{1}{h_1} \frac{\partial h_1}{\partial y_2}$$

The critical equation of motion is then the first, which has the added term $\frac{1}{h_1} \frac{\partial S_{11}}{\partial y_1}$ that Higashitani and Pritchard assume to be zero. In undisturbed flow this term is $\sigma_{11,1}$, but eq. (9) shows $\sigma_{11,1} = -k$ in undisturbed plane Poiseuille flow. According to eq. (16) and our observation about eq. (19), were the HPBL derivation correct, a substantial contribution to the integral must come from the upper half of the channel, where the flow is nearly undisturbed, and the pressure gradient is nearly its undisturbed value. It therefore seems unlikely that $\frac{1}{h_1} \frac{\partial S_{11}}{\partial y_1}$ can be ignored. Leaving this term in and continuing as Higashitani and Pritchard do implies the result of Flaw C.

The derivation of Appendix II would need some extra arguing to apply directly to the case of undisturbed flow but appears to be valid in that case nevertheless, since the inclusion of the missing term resolves the anomaly:

$$\sigma_{22}(h/2) - \sigma_{22}(0) = -\frac{1}{2} \int_{y_2(0)}^{y_2(h/2)} \frac{N_1}{\tau} \left(\frac{1}{h_2} \frac{\partial \tau}{\partial y_2} - k \right) h_2 dy_2 \quad (24)$$

And in this flow

$$\frac{1}{h_2} \frac{\partial \tau}{\partial y_2} - k = \sigma_{12,2} - k = 0$$

From which the desired result follows. Note that the key to the validity of the result is including the term omitted in the derivation of ref. 1.

Except in the undisturbed plane Poiseuille case, the new integral does not seem amenable to a change of variable to shear stress. Furthermore, the presence of the material derivative terms, the poles, multiple-valued nature of the integrand seem to render it intractable, or even nonexistent. The best that can be said is that for a second order fluid, the relation reduces to $-N_1/4$.

6. Conclusions

The HPBL relation does not seem to be applicable to plane Poiseuille flow. Its prediction seems correct in a second order fluid, but even there its derivation is suspect. However, a subsequent report [7] will demonstrate that numerical results uphold the relation

$$\frac{N_1}{2\sigma} = \frac{dP_e}{d\sigma} \quad (25)$$

as a reasonably accurate measurement relation. This is the relation with practical measurement value, and though it can be formally derived via HPBL, the authors believe an independent verification (possibly numerical) or analytic derivation would provide a more convincing testimony to its worth than the current derivation.

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APPENDIX I : NOTATION

- A** - First Rivlin-Ericksen tensor, matrix notation.
- B** - Second Rivlin-Ericksen tensor, matrix notation.
- b* - Slot width.
- C* - Constant of integration.
- \mathcal{C} - Centerline of slot.
- d* - Slot depth.
- D_e - Deborah number $\equiv (T - \Lambda)\dot{\gamma}_w$.
- $\frac{D}{Dt}$ - Material derivative.
- f* - Arbitrary scalar function.
- g* - Arbitrary scalar function.
- h* - Die height.
- h_i - Scale factors for streamline co-ordinates.
- I** - Identity tensor, matrix notation.
- k* - Streamwise pressure gradient.
- L* - Length of Ω in x_1 -direction.
- Λ - Retardation time.
- $N_1 - \sigma_{11} - \sigma_{22} = S_{11} - S_{22}$.

- N_{1w} - N_1 at wall in undisturbed flow.
 N_{1w}^0 - N_1 at wall opposing slot.
 $N_2 - \sigma_{22} - \sigma_{33} = S_{22} - S_{33}$
 p - Hydrostatic pressure function.
 \bar{p} - p in Stokes-flow.
 P - Thrust, $-\sigma_{22}$.
 P_1 - P at wall opposing slot.
 P_2 - P at slot bottom.
 $P_e = P_1 - P_2$.
 $q = L/2 - b/2$.
 S_{ij} - Stress tensor in streamline system (physical components).
 T - Typical relaxation time.
 T^* - Second-order fluid parameter determining N_2/N_1 .
 u, u_i - Velocity vector, rheological co-ordinates.
 u_{max} - u_1 at channel center in undisturbed flow.
 x_i - Rheological co-ordinates.
 x_2^a, x_2^b - Points on \mathbb{C} .
 y_i - Streamline co-ordinates.
 y_2^a, y_2^b - Streamline co-ordinates of x_2^a and x_2^b .
 $\frac{\partial}{\partial y_i}$ - Partial differentiation with respect to streamline co-ordinates.
 $\dot{\gamma} = \pm(tr A^2/2)^{1/2}$.
 $\dot{\gamma}_w$ - $\dot{\gamma}$ at wall in undisturbed flow.
 $\dot{\gamma}_w^0$ - $\dot{\gamma}$ at wall opposing slot.
 η - Viscosity.
 ν_1 - First normal-stress coefficient.
 Ω - Problem domain, truncated channel of length L , with or without a slot.
 σ, σ_{ij} - Stress tensor, rheological co-ordinates,
 $\tau = \sigma_{12} = S_{12}$ on \mathbb{C} .
 $\sigma_w = \sigma_{12}$ at wall in undisturbed flow.
 $\sigma_w^0 = \tau$ at wall opposing slot.
 $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ - Christoffel symbols for streamline co-ordinates.
 (\cdot) , - Partial differentiation with respect to rheological co-ordinates.

APPENDIX II : MODIFIED HPBL DERIVATION

Equations of motion in streamline coordinates

$$\frac{h_\alpha}{h_1 h_2 h_3} \frac{\partial}{\partial y_j} \left(\frac{h_1 h_2 h_3}{h_\alpha h_j} S_{j\alpha} \right) + \left\{ \begin{smallmatrix} \alpha \\ jk \end{smallmatrix} \right\} \frac{h_\alpha}{h_j h_k} S_{jk} = 0 \quad \alpha = 1, 2$$

H-P assumption: $\frac{\partial h_i}{\partial y_j} = 0$ except when $i = 1, j = 2$. By definition

$$[g] = \begin{bmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left\{ \begin{matrix} i \\ \alpha\beta \end{matrix} \right\} = \frac{1}{2g_{i\sigma}} \left(\frac{\partial g_{\sigma\beta}}{\partial y_\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial y_\beta} - \frac{\partial g_{\alpha\beta}}{\partial y_\sigma} \right)$$

so

$$\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} = \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} = 0$$

$$\left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} = -\frac{h_1}{h_2^2} \frac{\partial h_1}{\partial y_2}$$

$$\left\{ \begin{matrix} 1 \\ 21 \end{matrix} \right\} = \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} = \frac{1}{h_1} \frac{\partial h_1}{\partial y_2}$$

For $\alpha = 1$:

$$\begin{aligned} & \frac{h_1}{h_1 h_2} \frac{\partial}{\partial y_j} \left(\frac{h_1 h_2}{h_1 h_j} S_{j1} \right) + \left\{ \begin{matrix} 1 \\ 21 \end{matrix} \right\} \frac{h_1}{h_2 h_1} S_{21} + \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} \frac{h_1}{h_1 h_2} S_{12} \\ &= \frac{1}{h_2} \frac{\partial}{\partial y_1} \left(\frac{h_2}{h_1} S_{11} \right) + \frac{1}{h_2} \frac{\partial}{\partial y_2} \left(\frac{h_2}{h_2} S_{21} \right) + \frac{2}{h_1 h_2} S_{21} \\ &= \frac{1}{h_1} \frac{\partial S_{11}}{\partial y_1} + \frac{1}{h_2} \frac{\partial S_{21}}{\partial y_2} + \frac{2}{h_1 h_2} \frac{\partial h_1}{\partial y_2} S_{21} = 0 \end{aligned}$$

For $\alpha = 2$:

$$\begin{aligned} & \frac{h_2}{h_1 h_2} \frac{\partial}{\partial y_j} \left(\frac{h_1 h_2}{h_2 h_j} S_{j2} \right) + \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} \frac{h_2}{h_1^2} S_{11} \\ &= \frac{1}{h_1} \frac{\partial}{\partial y_1} (S_{12}) + \frac{1}{h_1} \frac{\partial}{\partial y_2} \left(\frac{h_1}{h_2} S_{22} \right) - \frac{h_1}{h_2^2} \frac{\partial h_1}{\partial y_2} \frac{h_2}{h_1^2} S_{11} \\ &= \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y_2} S_{22} + \frac{1}{h_2} \frac{\partial S_{22}}{\partial y_2} - \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y_2} S_{11} \end{aligned}$$

or

$$\frac{1}{h_1} \frac{\partial h_1}{\partial y_2} = \frac{\frac{\partial S_{22}}{\partial y_2}}{(S_{11} - S_{22})}$$

and finally

$$\frac{\partial S_{22}}{\partial y_2} = -\frac{S_{11} - S_{22}}{2S_{12}} \left(\frac{1}{h_1} \frac{\partial S_{11}}{\partial y_1} + \frac{1}{h_2} \frac{\partial S_{21}}{\partial y_2} \right) h_2$$

Integrating along \mathcal{C}

$$P_e = \int_{y_2^*}^{y_2^b} \left(\frac{1}{h_1} \frac{\partial S_{11}}{\partial y_1} + \frac{1}{h_2} \frac{\partial S_{21}}{\partial y_2} \right) \frac{N_1}{2S_{12}} h_2 dy_2$$

Note that in undisturbed flow on \mathcal{C} $\frac{1}{h_1} \frac{\partial S_{11}}{\partial y_1}$ is the Cartesian covariant derivative so that $\frac{1}{h_1} \frac{\partial S_{11}}{\partial y_1} = -k$.

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